Introduction to computer vision III

Instructors: Jean Ponce and Matthew Trager
jean.ponce@inria.fr, matthew.trager@cims.nyu.edu

TAs: Jiachen (Jason) Zhu and Sahar Siddiqui
jiachen.zhu@nyu.edu, ss12414@nyu.edu

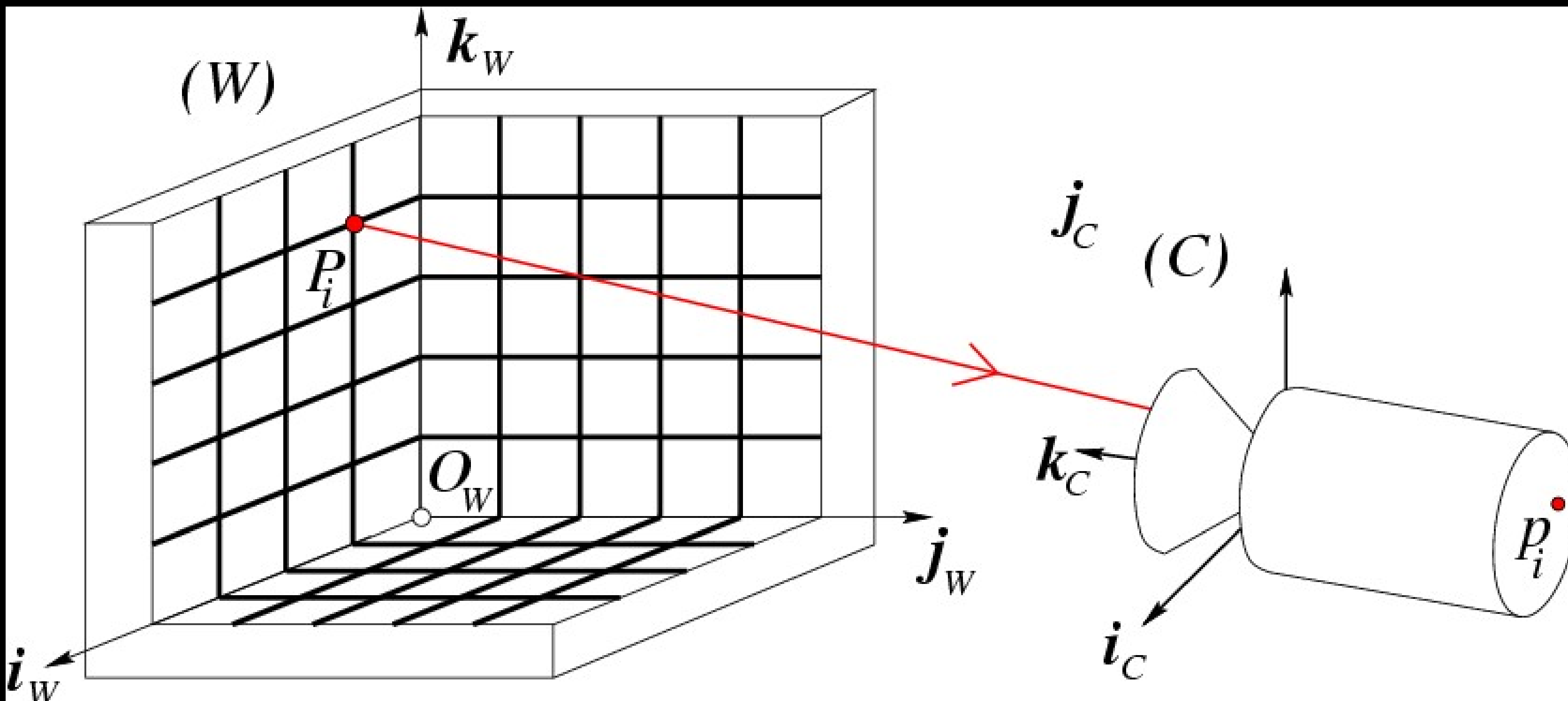
Slides will be available after class at:
https://mtrager.github.io/introCV-fall2019/

First homework is there!

Camera geometry and calibration III

- Intrinsic and extrinsic parameters
- Strong (Euclidean) calibration
- Degenerate configurations
- What about affine cameras?

Quantitative Measurements and Calibration



Euclidean Geometry

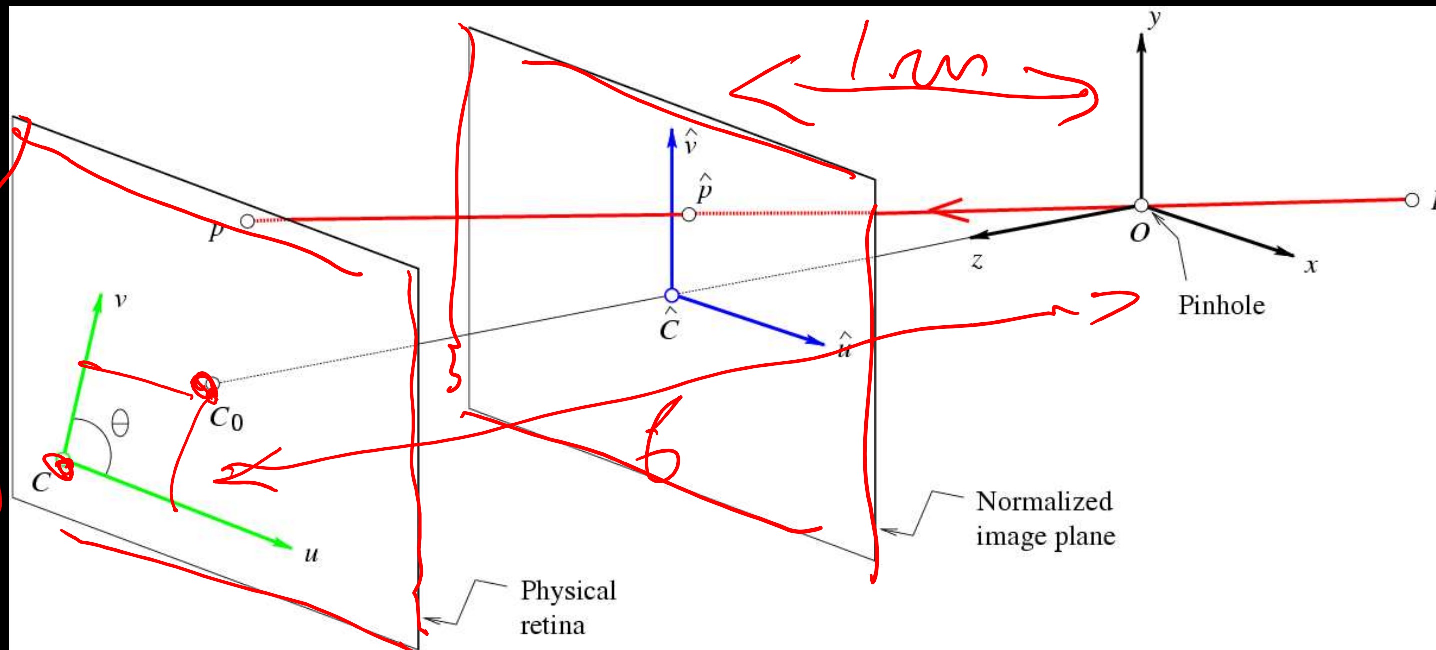
The intrinsic parameters of a camera

Units:

k,l : pixel/m

f : m

α,β : pixel



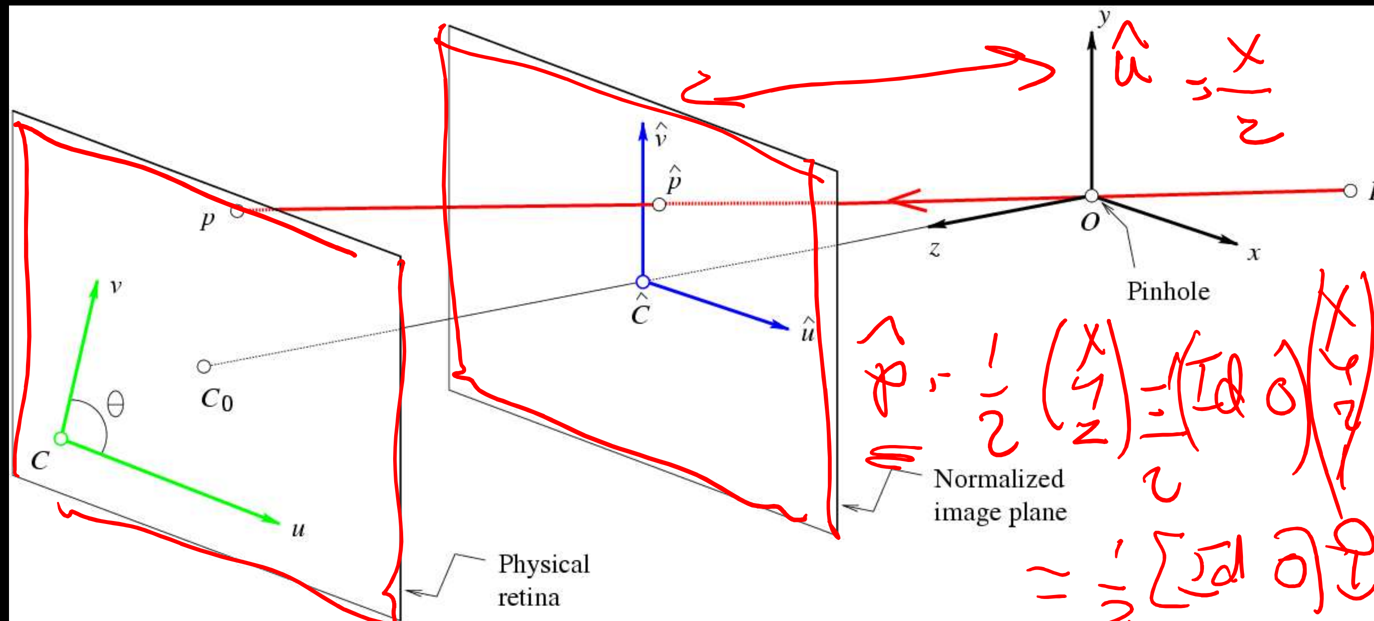
$$\begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases} \Longleftrightarrow \hat{\boldsymbol{p}} = \frac{1}{z}\left(\mathrm{Id} \quad \mathbf{0}\right)\begin{pmatrix} \boldsymbol{P} \\ 1 \end{pmatrix}$$

Normalized image coordinates

Physical image coordinates

$$\begin{cases} u = kf\dfrac{x}{z} \\ v = lf\dfrac{y}{z} \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} + u_0 \\ v = \beta\dfrac{y}{z} + v_0 \end{cases} \rightarrow \begin{cases} u = \alpha\overset{\hat{u}}{\dfrac{x}{z}} - \alpha\cot\theta\overset{\hat{v}}{\dfrac{y}{z}} + u_0 \\ v = \dfrac{\beta}{\sin\theta}\overset{\hat{v}}{\dfrac{y}{z}} + v_0 \end{cases}$$

The intrinsic parameters of a camera



Calibration matrix

$$\boldsymbol{p} = \mathcal{K}\hat{\boldsymbol{p}}, \quad \text{where} \quad \boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{K} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

Homogeneous coordinates

The perspective projection equation

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \mathcal{M} \stackrel{\text{def}}{=} \left(\mathcal{K} \quad \boldsymbol{0}\right)$$

The Extrinsic Parameters of a Camera

- When the camera frame (C) is different from the world frame (W),

$$\begin{pmatrix} {}^C P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^C_W\mathcal{R} & {}^C O_W \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}.$$

$$p = \frac{1}{z} M\, {}^C P$$

$${}^C P = R\, {}^W P + t$$

$$= {}^C_W R\, {}^W P + {}^C O_W$$

$$\begin{bmatrix} {}^C P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^C_W R & {}^C O_W \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} {}^W P \\ 1 \end{bmatrix}$$

$$0^T = (0,0,0)$$

- Thus,

$$p = K[\mathrm{id}\ \ 0]\, {}^C P$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P},} \quad \text{where} \quad \begin{cases} \mathcal{M} = \mathcal{K}(\mathcal{R} \quad \boldsymbol{t}), \\ \mathcal{R} = {}^C_W\mathcal{R}, \\ \boldsymbol{t} = {}^C O_W, \\ \boldsymbol{P} = \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}. \end{cases}$$

$$M = K[R\ \ t]$$

$$p = \frac{1}{z} M P$$

- Note: z is *not* independent of $\mathcal{M}$ and $\boldsymbol{P}$:

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{pmatrix} \Longrightarrow z = \boldsymbol{m}_3 \cdot \boldsymbol{P}, \quad \text{or} \quad \begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}, \\ v = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}. \end{cases}$$

Explicit Form of the Projection Matrix

$$\lambda\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

$\lambda \neq 0$

Note: If $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ then $|\boldsymbol{a}_3| = 1$.

Replacing $\mathcal{M}$ by $\lambda\mathcal{M}$ in

$K = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$ $\quad t = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$

$$\begin{cases} u = \dfrac{\cancel{\lambda}\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\cancel{\lambda}\boldsymbol{m}_3 \cdot \boldsymbol{P}} \\ v = \dfrac{\cancel{\lambda}\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\cancel{\lambda}\boldsymbol{m}_3 \cdot \boldsymbol{P}} \end{cases}$$

does not change u and v.

M is only defined up to scale in this setting!!

Explicit Form of the Projection Matrix

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

$a_1^T =$

$a_2^T =$

$a_3^T = \boldsymbol{r}_3^T$

Can any 3x4 matrix be written that way?

2 to show $(\theta = \frac{\pi}{2})$

$$M = \begin{bmatrix} a_1^T & b_1 \\ a_2^T & b_2 \\ a_3^T & b_3 \end{bmatrix}$$

$$\begin{cases} a_1 = \alpha r_1 + u_0 r_3 \\ a_2 = \beta r_2 + v_0 r_3 \\ a_3 = r_3 \end{cases}$$

$(a_1 \times a_3) = \alpha \cancel{\beta r_3 - \alpha v_0 r_2 - u_0 \beta} r_1$

$(a_2 \times a_3) = \beta r_1$

Theorem (Faugeras, 1993)

Let $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ be a 3×4 matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

Geometric Interpretation

Projection equation:

$$u = \frac{m_1^T P}{m_3^T P} = \frac{a_1^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + b_1}{a_3^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + b_3}$$

Observations:

- $$a_1^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + b_1 = 0$$

  is the equation of a plane of normal direction a_1

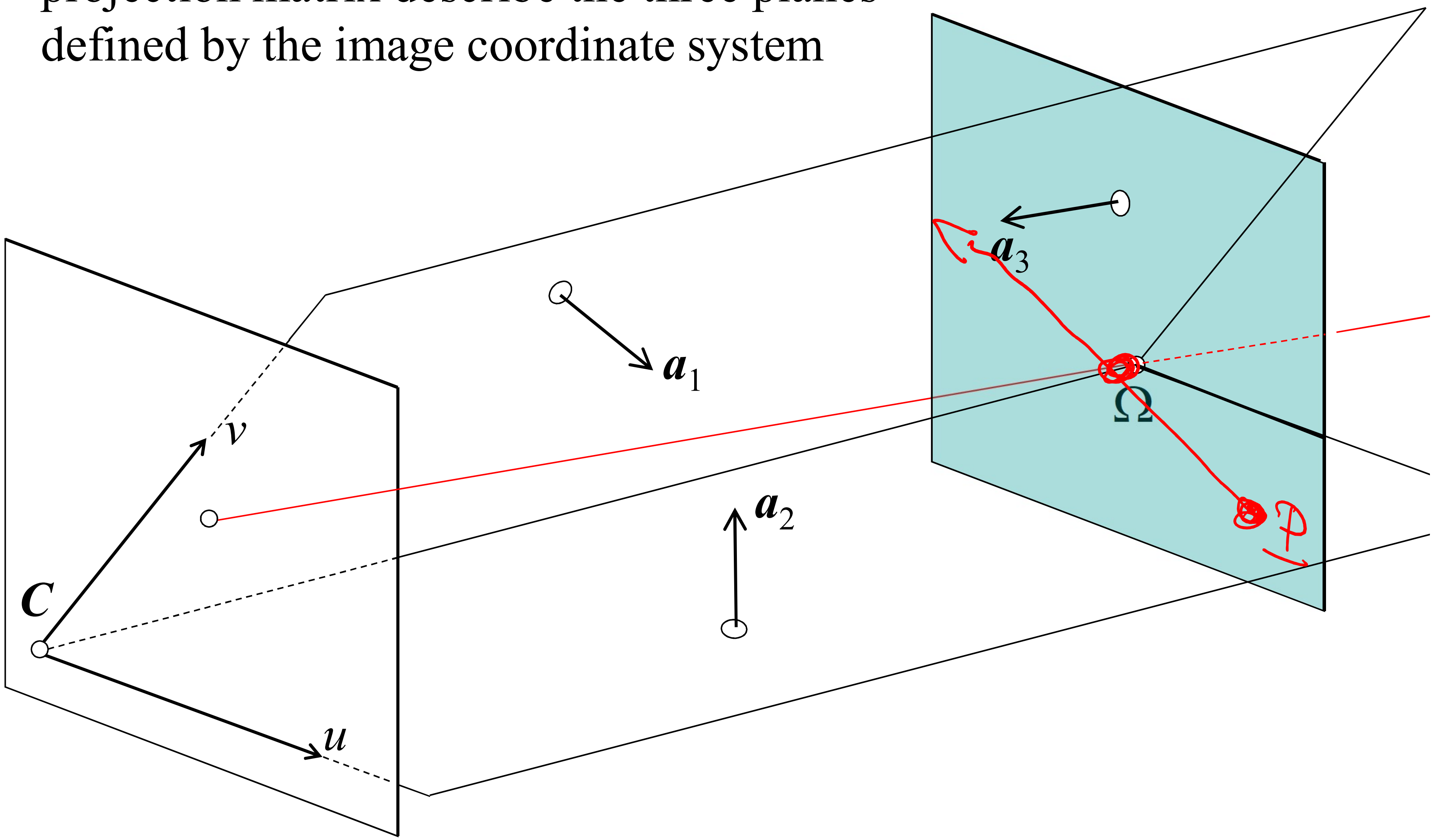
- From the projection equation, it is also the plane defined by: $u = 0$

- Similarly:
  - (a_2, b_2) describes the plane defined by: $v = 0$
  - (a_3, b_3) describes the plane defined by:

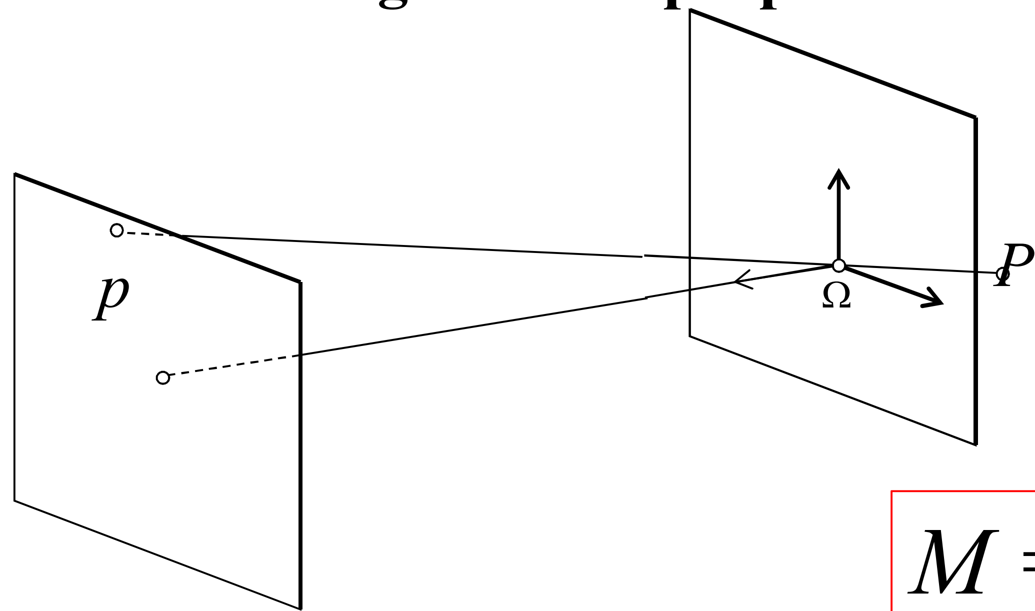
    $$u = \infty \quad v = \infty$$

    → That is the plane passing through the pinhole ($z = 0$)

Geometric Interpretation: The rows of the projection matrix describe the three planes defined by the image coordinate system



Other useful geometric properties



$$\boxed{M = [A, b]}$$

Q: Given an image point *p*, what is the direction of the corresponding ray in space?

A: $$\boxed{w = A^{-1} p}$$

$$p = \frac{1}{z} M \begin{bmatrix} P \\ 1 \end{bmatrix} = \frac{1}{z}[AP + b]$$

Q: Can we compute the position of the camera center Ω?

A: $$\boxed{\Omega = -A^{-1} b}$$

Linear Camera Calibration

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$u = \frac{m_1 \cdot P}{m_3 \cdot P} \qquad v = \frac{m_2 \cdot P}{m_3 \cdot P}$$

$$\Longleftrightarrow \begin{bmatrix} P^T & 0^T & -uP^T \\ 0^T & P^T & -vP^T \end{bmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0$$

$$\underbrace{\begin{bmatrix} P_1^T & 0^T & -u_1 P_1^T \\ 0^T & P_1^T & -v_1 P_1^T \\ \ldots & & \\ P_n^T & 0^T & -u_n P_n^T \\ 0^T & P_n^T & -v_n P_n^T \end{bmatrix}}_{\mathcal{P}\ 2n \times 12} \underbrace{\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}}_{\in \mathbb{R}^{12}} = 0$$

$$(0^T = (0,0,0,0))$$

$$\Longleftrightarrow \boxed{\mathcal{P} m = 0}$$

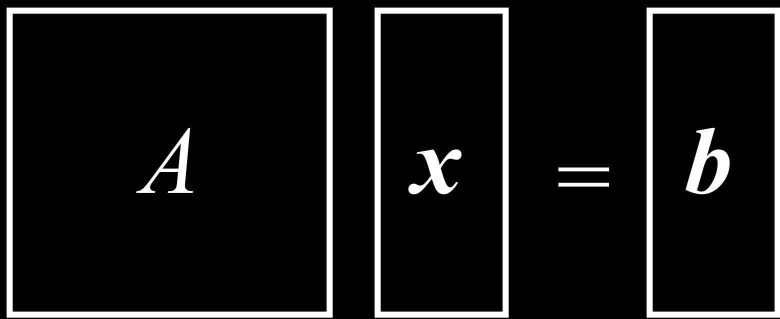
Linear Camera Calibration

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \iff \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

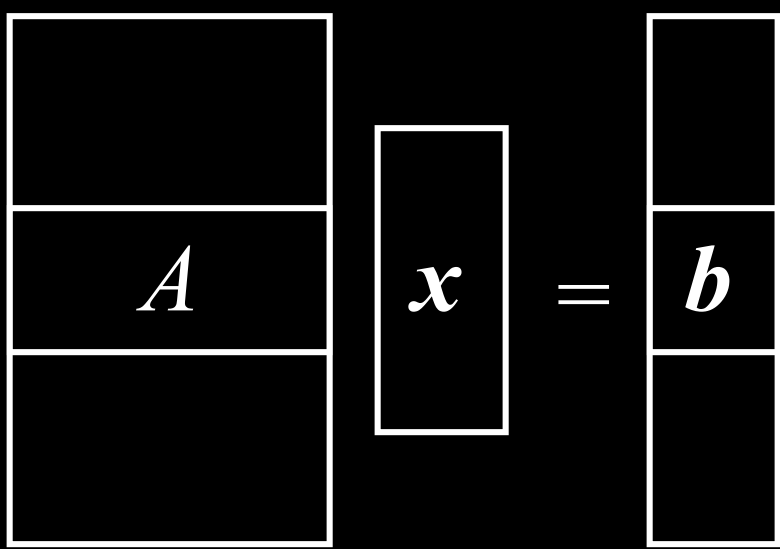
$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \ldots & \ldots & \ldots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

Linear Systems



Square system:

- unique solution
- Gaussian elimination



Rectangular system ??

- underconstrained: infinity of solutions
- overconstrained: no solution

Minimize $||Ax-b||^2$

How do you solve overconstrained linear equations ??

$\min_x \|Ax-b\|^2 \quad E = \|Ax-b\|^2 = e \cdot e$, with $e = Ax-b$

$$\frac{\partial E}{\partial x} = 0 = \frac{\partial}{\partial x}(e \cdot e) = 2\frac{\partial e}{\partial x} \cdot e \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad A = [c_1 \dots c_m]$$

$$\frac{\partial e}{\partial x_i} = \left[\frac{\partial}{\partial x_i}(Ax-b)\right] \cdot (Ax-b)$$

$$= \left[\frac{\partial}{\partial x_i} \underbrace{(x_1 c_1 + \dots + x_m c_m - b)}_{c_i}\right] \cdot (Ax-b) = 0$$

$$\begin{cases} c_1 \cdot (Ax-b) = 0 \\ \vdots \\ c_m \cdot (Ax-b) = 0 \end{cases} \iff \underbrace{\begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix}}_{A^T} (Ax-b) = 0$$

$$A^T(Ax-b) = 0$$

$$A^T A x = A^T b$$

$$x = \underbrace{(A^T A)^{-1} A^T}_{\text{pseudo inverse}} b$$

How do you solve overconstrained linear equations ??

- Define $E = |\boldsymbol{e}|^2 = \boldsymbol{e} \cdot \boldsymbol{e}$ with

$$\begin{aligned} \boldsymbol{e} &= A\boldsymbol{x} - \boldsymbol{b} = \left[\begin{array}{c|c|c|c} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \dots & \boldsymbol{c}_n \end{array} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \boldsymbol{b} \\ &= x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots x_n\boldsymbol{c}_n - \boldsymbol{b} \end{aligned}$$

- At a minimum,

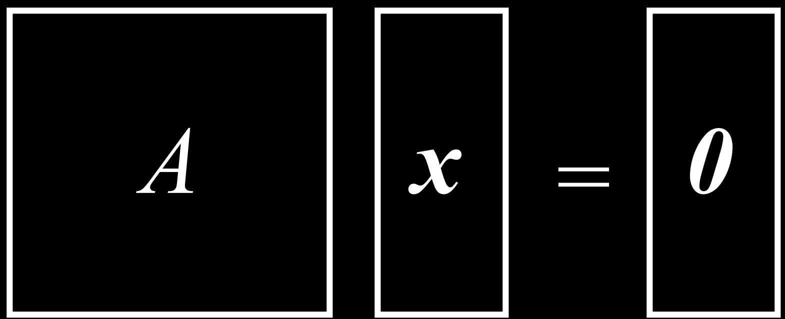
$$\begin{aligned} \frac{\partial E}{\partial x_i} &= \frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{e}}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} \\ &= 2\frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n - \boldsymbol{b}) \cdot \boldsymbol{e} = 2\boldsymbol{c}_i \cdot \boldsymbol{e} \\ &= 2\boldsymbol{c}_i^T(A\boldsymbol{x} - \boldsymbol{b}) = 0 \end{aligned}$$

- or

$$0 = \begin{bmatrix} \boldsymbol{c}_i^T \\ \vdots \\ \boldsymbol{c}_n^T \end{bmatrix} (A\boldsymbol{x} - \boldsymbol{b}) = A^T(A\boldsymbol{x} - \boldsymbol{b}) \Rightarrow A^TA\boldsymbol{x} = A^T\boldsymbol{b},$$

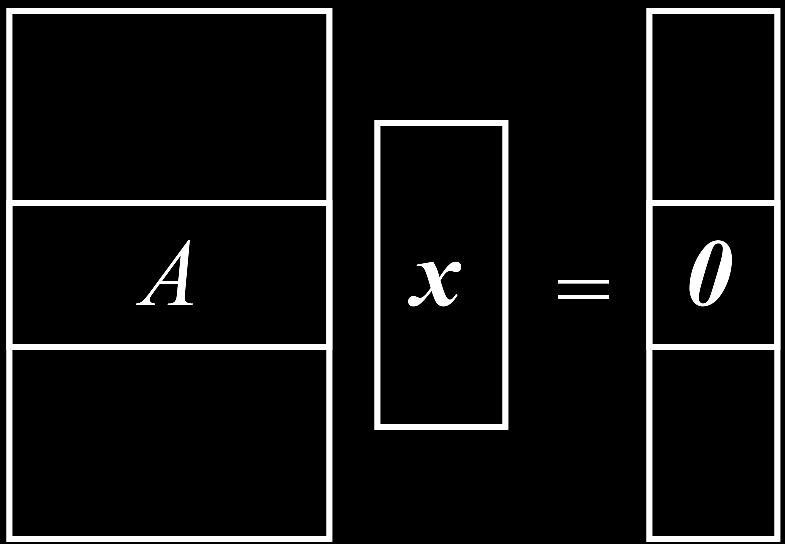
where $\boldsymbol{x} = A^\dagger\boldsymbol{b}$ and $A^\dagger = (A^TA)^{-1}A^T$ is the *pseudoinverse* of A !

Homogeneous Linear Systems



Square system:

- unique solution: 0
- unless Det(A)=0



Rectangular system ??

- 0 is always a solution

Minimize $||Ax||^2$
under the constraint
$||x||^2 = 1$

How do you solve overconstrained homogeneous linear equations ??

$\min_x \|Ax\|^2$ under $\|x\|^2 = 1$

$\|Ax\|^2 = x^T U x \quad U = A^T A, \quad \|Ax\|^2 = (Ax)\cdot(Ax) = (Ax)^T (Ax)$

$= (x^T A^T)(Ax) = x^T (A^T A) x$

$e_1, \dots e_n$ the eigenvectors of U

$\lambda_1, \dots \lambda_n$ the associated eigenvalues

$0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$

$$\boxed{\begin{array}{c}\operatorname{Argmin}_{\|x\|^2=1} \|Ax\|^2 = e_1\end{array}}$$

$x = \mu_1 e_1 + \dots + \mu_n e_n$

$Ux = \mu_1 \lambda_1 e_1 + \dots + \mu_n \lambda_n e_n$

$x^T U x = (\mu_1 e_1 + \dots + \mu_n)\cdot(\lambda_1 \mu_1 e_1 + \dots + \lambda_n \mu_n e_n) = \lambda_1 \mu_1^2 + \dots + \lambda_n \mu_n^2$

$\ge \lambda_1 (\mu_1^2 + \dots + \mu_n^2) = \lambda_1$

How do you solve overconstrained homogeneous linear equations ??

$$E = |\mathcal{U}\boldsymbol{x}|^2 = \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x}$$

- Orthonormal basis of eigenvectors: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q$.
- Associated eigenvalues: $0 \leq \lambda_1 \leq \ldots \leq \lambda_q$.
- Any vector can be written as

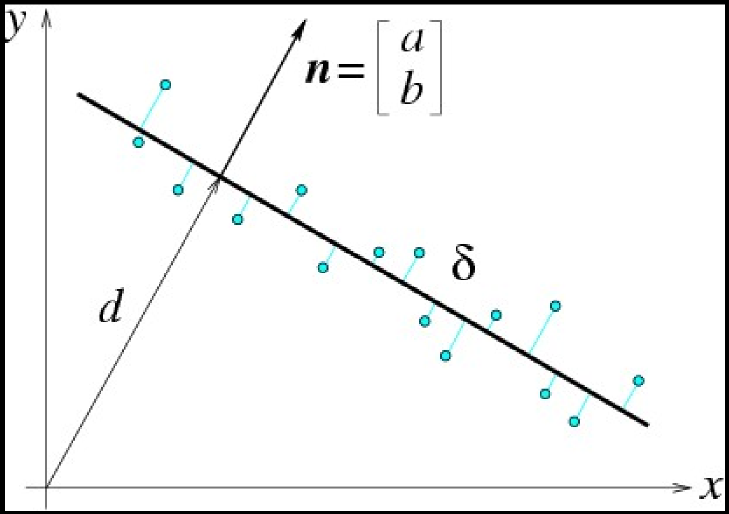
$$\boldsymbol{x} = \mu_1\boldsymbol{e}_1 + \ldots + \mu_q\boldsymbol{e}_q$$

for some μ_i $(i = 1, \ldots, q)$ such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$\begin{aligned} E(\boldsymbol{x})-E(\boldsymbol{e}_1) &= \boldsymbol{x}^T(U^TU)\boldsymbol{x}-\boldsymbol{e}_1^T(U^TU)\boldsymbol{e}_1 \\ &= \lambda_1\mu_1^2+ \ldots +\lambda_q\mu_q^2-\lambda_1 \\ &\geq \lambda_1(\mu_1^2+ \ldots +\mu_q^2-1)=0 \end{aligned}$$

The solution is $\boldsymbol{e}_1$.

Example: Line Fitting



Problem: minimize

$$E(a,b,d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

with respect to (a,b,d).

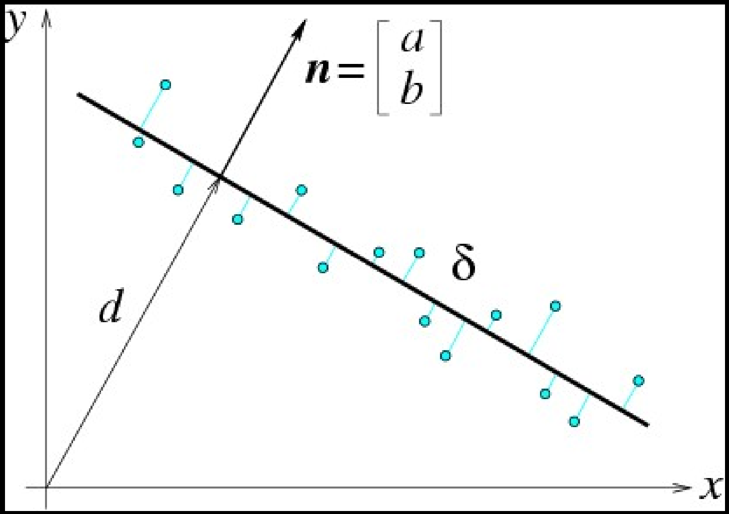
$$\frac{\partial E}{\partial d} = \sum_{i=1}^{m} -2(ax_i + by_i - d) = 0 \iff d = a\bar{x} + b\bar{y}$$

$$\bar{x} = \frac{1}{m}\sum x_i, \quad \bar{y} = \frac{1}{m}\sum y_i$$

$$E = \sum_{i=1}^{m} (ax + by - d)^2$$

$$= \sum_{i=1}^{m} [a(x_i - \bar{x}) + b(y_i - \bar{y})]^2 \Longrightarrow \min_{\|a,b\|^2=1} \left\| A \begin{pmatrix} a \\ b \end{pmatrix} \right\|^2$$

Example: Line Fitting



Problem: minimize

$$E(a,b,d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

with respect to (*a*,*b*,*d*).

- Minimize *E* with respect to *d*:

$$\frac{\partial E}{\partial d} = 0 \Longrightarrow d = \sum_{i=1}^{n} \frac{ax_i + by_i}{n} = a\bar{x} + b\bar{y}$$

- Minimize *E* with respect to *a*,*b*:

$$E = \sum_{i=1}^{n} [a(x_i - \bar{x}) + b(y_i - \bar{y})]^2 = |\mathcal{U}\boldsymbol{n}|^2$$ where $$\mathcal{U} = \begin{pmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \dots & \dots \\ x_n - \bar{x} & y_n - \bar{y} \end{pmatrix}$$

- Done !!

Note:

$$\mathcal{U}^T\mathcal{U} = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 & \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \\ \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} & \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \end{pmatrix}$$

- Matrix of second moments of inertia

- Axis of least inertia

Linear Camera Calibration

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \ldots & \ldots & \ldots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

Minimize $||Pm||^2$ under the constraint $||m||^2 = 1$

Once M is known, you still got to recover the intrinsic and extrinsic parameters !!!

This is a decomposition problem, not an estimation problem.

$$\rho\,\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

- Intrinsic parameters
- Extrinsic parameters

Degenerate Point Configurations

Are there other solutions besides *M* ??

$$\mathbf{0} = \mathcal{P}\boldsymbol{l} = \begin{pmatrix} \boldsymbol{P}_1^T & \mathbf{0}^T & -u_1\boldsymbol{P}_1^T \\ \mathbf{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \dots & \dots & \dots \\ \boldsymbol{P}_n^T & \mathbf{0}^T & -u_n\boldsymbol{P}_n^T \\ \mathbf{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1^T\boldsymbol{\lambda} - u_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \boldsymbol{P}_1^T\boldsymbol{\mu} - v_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \dots \\ \boldsymbol{P}_n^T\boldsymbol{\lambda} - u_n\boldsymbol{P}_n^T\boldsymbol{\nu} \\ \boldsymbol{P}_n^T\boldsymbol{\mu} - v_n\boldsymbol{P}_n^T\boldsymbol{\nu} \end{pmatrix}$$

$$\begin{cases} P^T\lambda - u P^T\nu = 0 \\ P^T\mu - v P^T\nu = 0 \end{cases}$$

$$\begin{cases} u = \dfrac{m_1 \cdot P}{m_3 \cdot P} \\ v = \dfrac{m_2 \cdot P}{m_3 \cdot P} \end{cases}$$

$$(P\cdot\lambda)(m_3\cdot P) - (m_1\cdot P)(P\cdot\nu) = 0$$

$$P^T(\lambda m_3^T - m_1\nu^T)P = 0$$

$$(P\cdot\mu)(m_3\cdot P) - (m_2\cdot P)(P\cdot\nu) = 0 \iff \begin{cases} P^T(\lambda m_3^T - m_1\nu^T)P = 0 \\ P^T(\mu m_3^T - m_2\nu^T)P = 0 \end{cases}$$

Degenerate Point Configurations

Are there other solutions besides *M* ??

$$\mathbf{0} = \mathcal{P}\boldsymbol{l} = \begin{pmatrix} \boldsymbol{P}_1^T & \mathbf{0}^T & -u_1\boldsymbol{P}_1^T \\ \mathbf{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \dots & \dots & \dots \\ \boldsymbol{P}_n^T & \mathbf{0}^T & -u_n\boldsymbol{P}_n^T \\ \mathbf{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1^T\boldsymbol{\lambda} - u_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \boldsymbol{P}_1^T\boldsymbol{\mu} - v_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \dots \\ \boldsymbol{P}_n^T\boldsymbol{\lambda} - u_n\boldsymbol{P}_n^T\boldsymbol{\nu} \\ \boldsymbol{P}_n^T\boldsymbol{\mu} - v_n\boldsymbol{P}_n^T\boldsymbol{\nu} \end{pmatrix}$$

$$\Downarrow$$

$$\left\{ \begin{aligned} \boldsymbol{P}_i^T\boldsymbol{\lambda} - \frac{\boldsymbol{m}_1^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0 \\ \boldsymbol{P}_i^T\boldsymbol{\mu} - \frac{\boldsymbol{m}_2^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} \boldsymbol{P}_i^T(\boldsymbol{\lambda}\boldsymbol{m}_3^T - \boldsymbol{m}_1\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0 \\ \boldsymbol{P}_i^T(\boldsymbol{\mu}\boldsymbol{m}_3^T - \boldsymbol{m}_2\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0 \end{aligned} \right.$$

- Coplanar points: $(\lambda, \mu, \nu) = (\pi,0,0)$ or $(0,\pi,0)$ or $(0,0,\pi)$
- Points lying on the intersection curve of two quadric surfaces = straight line + twisted cubic

Does not happen for 6 or more random points!

Analytical Photogrammetry

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

Find $\boldsymbol{i}$ and $\boldsymbol{e}$ such that

$$\sum_{i=1}^{n}\left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e})\cdot\boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e})\cdot\boldsymbol{P}_i}\right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e})\cdot\boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e})\cdot\boldsymbol{P}_i}\right)^2\right] \quad \text{is minimized}$$

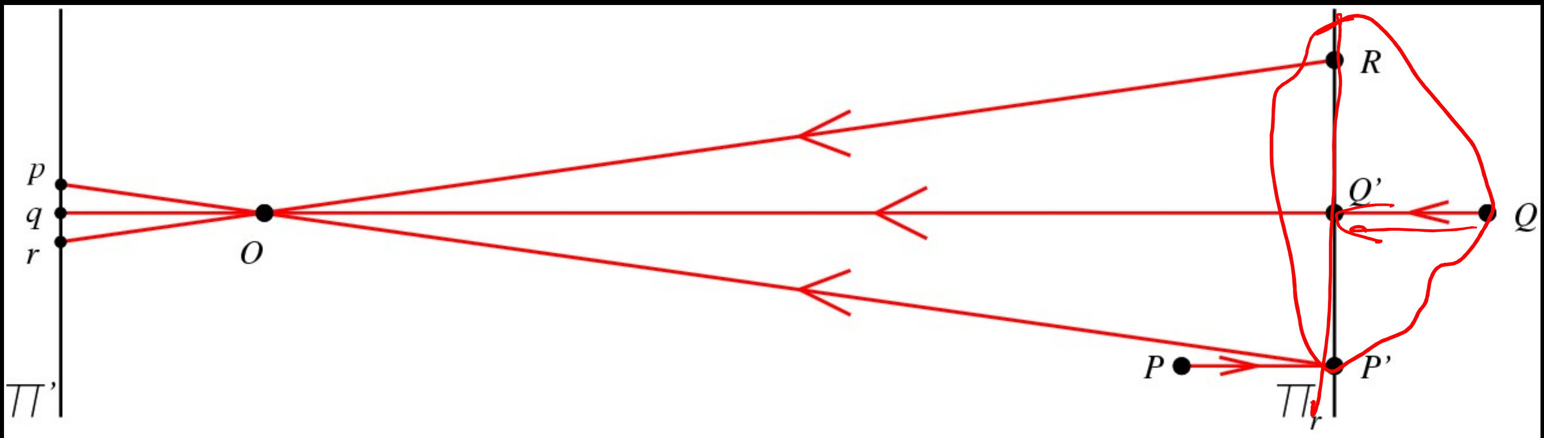
Non-Linear Least-Squares Methods

- Newton
- Gauss-Newton
- Levenberg-Marquardt

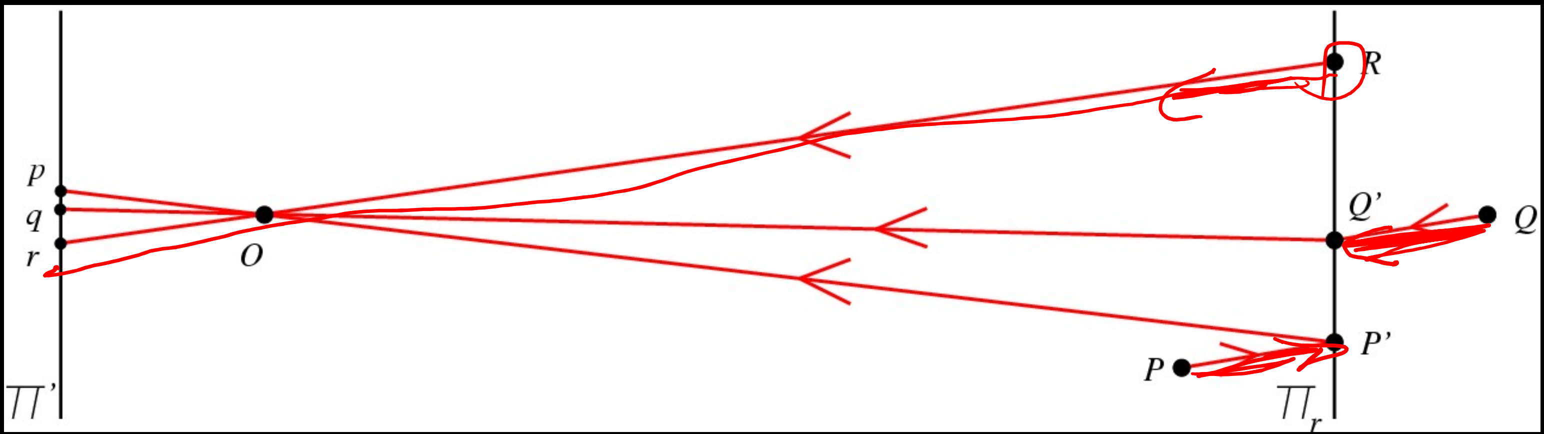
Iterative, quadratically convergent in favorable situations

What about Affine Cameras?

Weak-Perspective Projection

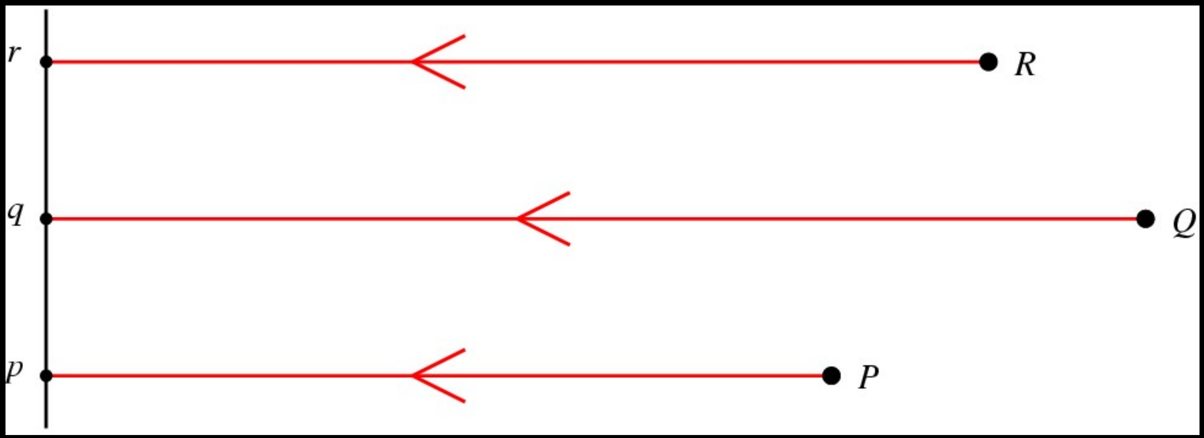


Paraperspective Projection

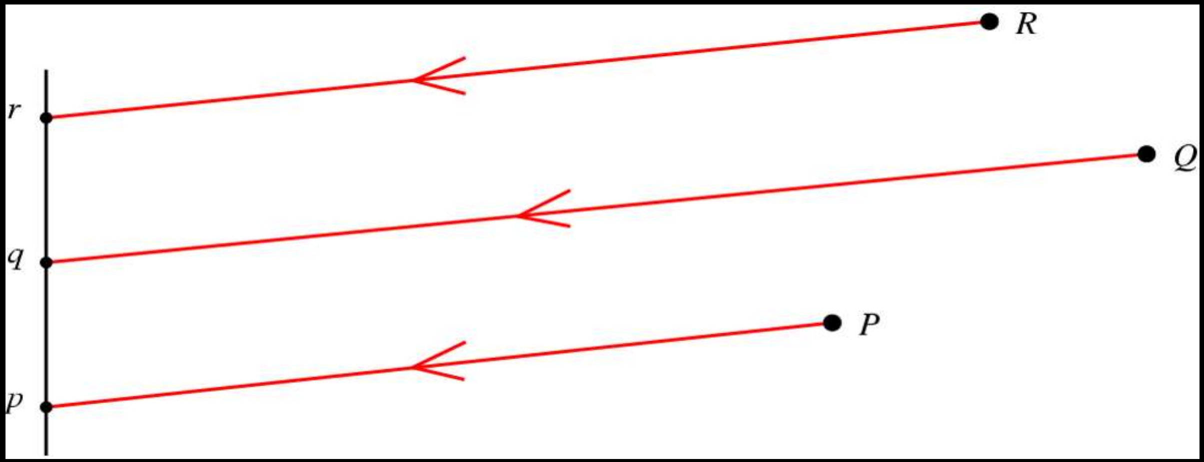


More Affine Cameras

Orthographic Projection



Parallel Projection



Weak-Perspective Projection Model

$$p = \frac{1}{z_r}\mathcal{M}P$$ (**p** and **P** are in homogeneous coordinates)

reference depth

$p = M\,P$ (**P** is in homogeneous coordinates)

$p = A\,P + b$ (neither **p** nor **P** is in hom. coordinates)

Definition: A 2x4 matrix **M** = [**A** **b**], where **A** is a rank-2 2x3 matrix, is called an affine projection matrix.

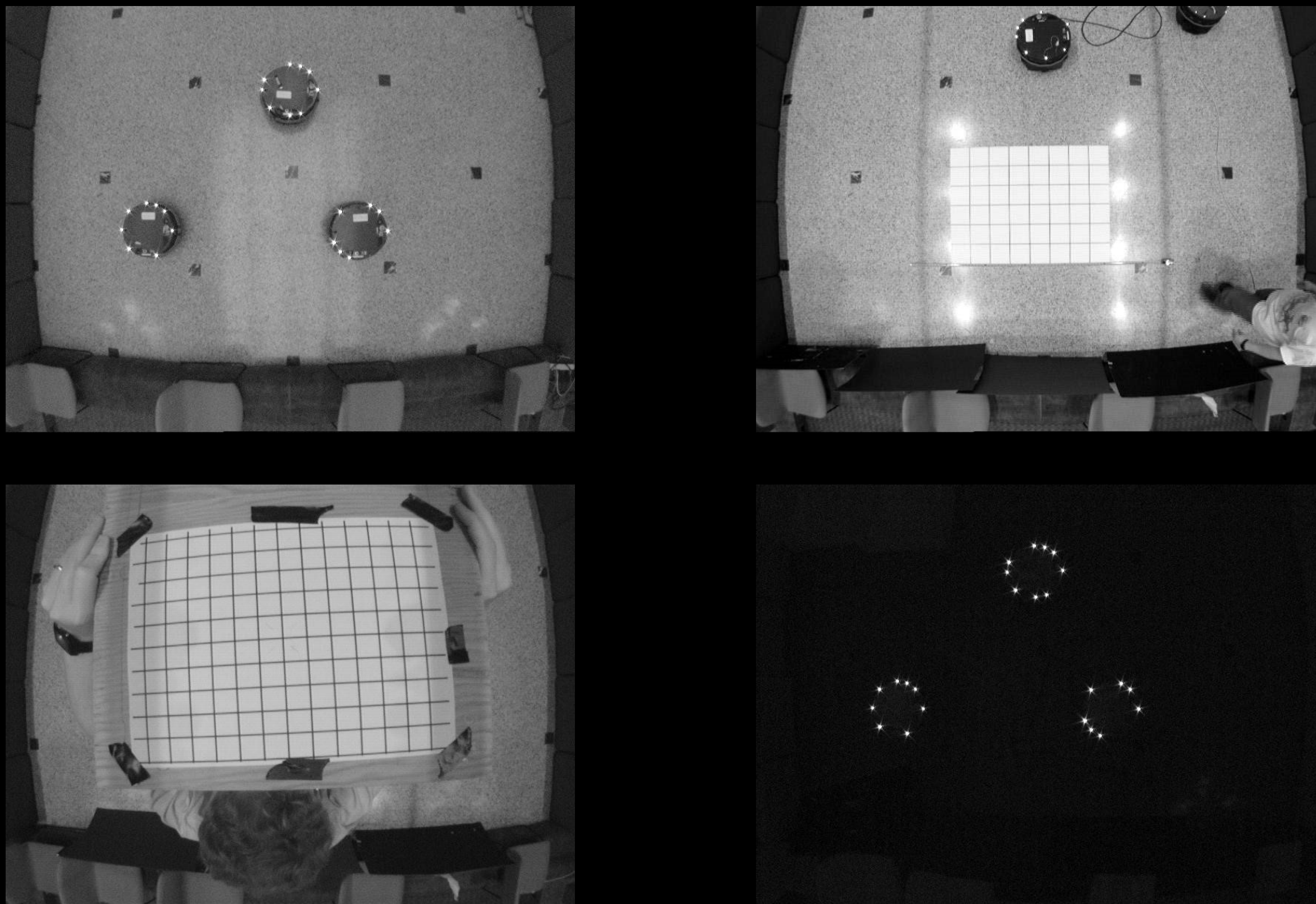
Theorem: All affine projection models can be represented by affine projection matrices.

General form of the weak-perspective projection equation:

$$\mathbf{M} = \frac{1}{z_r}\begin{bmatrix} k & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_2 & \mathbf{t}_2 \end{bmatrix} \qquad (1)$$

Theorem: An affine projection matrix can be written uniquely (up to a sign amibguity) as a weak perspective projection matrix as defined by (1).

Applications: Mobile Robot Localization (Devy et al., 1997)



(Rothganger, Sudsang, Ponce, 2002)